ON THE PROBLEM OF THE MOTION OF AN AXIALLY SYMMETRICAL BODY UNDER THE ACTION OF A CONSTANT MOMENT

(K ZADACHE DVIZHENIIA OSESIMMETRICHNOGO TELA POD DEISTVIEM POSTOIANNOGO MOMENTA)

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The paper uses continued fractions to study the motion of a solid, axially symmetrical body about a fixed point O when a constant moment acts along the axis of symmetry.

1. Consider a rectangular system of coordinates $\xi\eta\zeta$ rigidly attached to the body. Symmetry of the body is assumed about the ζ-axis, in which case the moments of inertia A and B about the axes ξ and η will be equal. A constant moment of magnitude m $(m > 0)$ is directed along the ζ-axis. The Euler equations for the projections ω_1, ω_2, ω_3 of the angular velocity ω on the moving axes of coordinates ξ, η, ζ are

$$A d\omega_1/dt + (C - A)\,\omega_2\omega_3 = 0, \qquad A d\omega_2/dt - (C - A)\,\omega_3\omega_1 = 0, \qquad C d\omega_3/dt = m \qquad (1.1)$$

and can be easily integrated [1](p.134). For initial conditions of a general form

$$\omega_1 = \omega_1^\circ, \quad \omega_2 = \omega_2^\circ, \quad \omega_3 = \omega_3^\circ, \quad t = 0, \quad (\omega_1^\circ)^2 + (\omega_2^\circ)^2 \neq 0 \qquad (1.2)$$

using the notation $i = \sqrt{-1}$, we have the solution [1]

$$\omega_1 + i\omega_2 = (\omega_1^\circ + i\omega_2^\circ)\exp\left(i\frac{C-A}{A}\int_0^t \omega_3\,dt\right), \qquad \omega_3 = \omega_3^\circ + \frac{m}{C}t \qquad (1.3)$$

We introduce a unit vector γ which retains a constant direction in space and we denote its projections on the moving axes of coordinates $\xi\eta\zeta$ by γ_1, γ_2, γ_3. These projections satisfy the equations [1](p.128)

$$d\gamma_1/dt = \omega_3\gamma_2 - \omega_2\gamma_3, \qquad d\gamma_2/dt = \omega_1\gamma_3 - \omega_3\gamma_1, \qquad d\gamma_3/dt = \omega_2\gamma_1 - \omega_1\gamma_3 \qquad (1.4)$$

Now consider a complex variable z [1](p.121)

$$z = (\gamma_1 + i\gamma_2)(1 - \gamma_3)^{-1} \qquad (1.5)$$

which defines completely the vector γ. If we differentiate z with respect to t on the basis of Equations (1.4) for z we obtain the Darboux-Riccati equation [1](p.130)

$$\frac{dz}{dt} = \frac{\omega_2 - i\omega_1}{2} - i\omega_3 z + \frac{\omega_2 + i\omega_1}{2}z^2 \qquad (1.6)$$

A change of variables of the form

$$u = z\frac{\omega_2 + i\omega_1}{|\omega_2 + i\omega_1|}, \qquad \tau = 0.5\,|\omega_2{}^\circ + i\omega_1{}^\circ|\left(t + \frac{\omega_3{}^\circ C}{m}\right) \tag{1.7}$$

leads to the differential equation

$$du/d\tau = 1 - i\alpha\tau u + u^2, \qquad \alpha = 4mA^{-1}\,|\omega_2{}^\circ + i\omega_1{}^\circ|^{-2} \tag{1.8}$$

If a particular solution of Equation (1.8) is known, then its solution reduces to quadratures. Equation (1.8) can be reduced to a linear differential equation of the second order [1](p.136).

Equation (1.8) describes a special case of motion of a body with angular velocities $\omega_1 = 0$, $\omega_2 = 2$, $\omega_3 = \alpha\tau$ when the variable τ is taken as time.

2. We seek a solution to Equation (1.8) by the method of Lagrange [2] (p.79). The substitution $u = t(1 - v)^{-1}$ leads to the differential equation

$$\tau\, dv/d\tau = (1 - \alpha i)\,\tau^2 - (1 - \alpha i\tau^2)\, v + v^2 \tag{2.1}$$

By replacing the independent variable τ^2 by x we transform (2.1) into the Riccati equation

$$2x\, dv\,/\,dx + (1 - i\alpha x)\, v - v^2 = (1 - i\alpha)\, x \tag{2.2}$$

We can find a particular solution to this equation in the form of a continued fraction [2](p.80), [3](p.295).

$$v = -\frac{(\alpha i - 1)\, x}{3} - \frac{(2\alpha i + 1)\, x}{5} + \frac{(3\alpha i - 1)\, x}{7} - \frac{(4\alpha i + 1)\, x}{9} + \cdots$$
$$\cdots - \frac{(2n\alpha i + 1)\, x}{4n + 1} + \frac{[(2n + 1)\,\alpha i - 1]\, x}{4n + 3} - \cdots \tag{2.3}$$

Taking into account the change of variables, we obtain the following particular solution to Equation (1.8)

$$u = \frac{\tau}{1} + \frac{(\alpha i - 1)\,\tau^2}{3} - \frac{(2\alpha i + 1)\,\tau^2}{5} + \frac{(3\alpha i - 1)\,\tau^2}{7} - \cdots \tag{2.4}$$

Using the notation of Pringsheim [2](p.8) we can write the solution (2.4) in the form

$$u = \left[\frac{\tau}{1},\ \frac{c_\nu \tau^2}{1}\right]_{\nu=2}^{\infty}, \qquad c_\nu = \frac{(-1)^\nu\,(\nu - 1)\,\alpha i - 1}{\nu^2 - 1} \tag{2.5}$$

Since $c_\nu \to 0$ as $\nu \to \infty$, the continued fraction in Expressions (2.4) and (2.5) for $u(\tau)$ converges for all finite values of τ (see [3] p.293). The solution obtained determines the position of the vector γ in the system of coordinates $\xi\eta\zeta$. This vector remains stationary in space and at the instant

$$t = -\,\omega_3{}^\circ\, Cm^{-1}, \qquad \tau = 0 \tag{2.6}$$

coincides in direction with the ζ-axis. The form of the solution is convenient for numerical computation but is not convenient for finding a general solution to (1.8) by means of quadratures.

3. Let us seek a general solution to Equation (1.8) with the initial conditions

$$u = b, \quad \tau = 0 \tag{3.1}$$

In Equation (1.8) we make a change of dependent variable

$$u = b\,(b - y)\,[b - y - (1 + b^2)\,\tau]^{-1} \tag{3.2}$$

We obtain the differential equation

$$b\tau\, dy\,/\,d\tau + (c + d\tau + e\tau^2)\, y + (-1 + f\tau)\, y^2 = g\tau + h\tau^2 \tag{3.3}$$

Here the constant coefficients are given by

$$c = b, \qquad d = -2 - 2i\alpha b^2 (1 + b^2)^{-1}, \qquad e = i\alpha b$$
$$f = i\alpha b (1 + b^2)^{-1}, \qquad g = -2b - i\alpha b^3 (1 + b^2)^{-1}, \ h = 1 + b^2 + i\alpha b^2 \tag{3.4}$$

Equation (3.3) is invariant in form with respect to a change of the type

$$y = g\tau (b + c - y_1)^{-1} \tag{3.5}$$

which reduces Equation (3.3) to Equation

$$b\tau\, dy_1 / d\tau + (c_1 + d_1\tau + e_1\tau^2) y_1 + (-1 + f_1\tau)\, y_1^2 = g_1\tau + h_1\tau^2 \tag{3.6}$$

The new coefficients are expressed in terms of the old by Formulas

$$c_1 = b + c, \qquad f_1 = -hg^{-1}, \qquad d_1 = -d - 2c_1 f_1, \qquad e_1 = -e$$
$$g_1 = g - dc_1 - f_1 c_1^2, \qquad h_1 = -gf - c_1 e \tag{3.7}$$

By making the change (3.5) repeatedly we obtain an expansion in a continued fraction. Eliminating the set of values b with a zero Lebesgue measure, we can construct a continued fraction with an infinite number of terms. The convergence of the resulting continued fractions has not been investigated.

4. For large values of τ we can employ a different method for finding a solution to Equation (1.8). We make the substitution

$$u = -\frac{dy}{d\tau} y^{-1} = -\frac{dy}{y d\tau} \tag{4.1}$$

which reduces (1.8) to a linear differential equation of the second order

$$\frac{d^2y}{d\tau^2} + i\alpha\tau \frac{dy}{d\tau} + y = 0 \tag{4.2}$$

Differentiating (4.2) k times with respect to τ, we find that

$$\frac{d^{k+2}y}{d\tau^{k+2}} + i\alpha\tau \frac{d^{k+1}y}{d\tau^{k+1}} + (1 + i\alpha k\tau) \frac{d^k y}{d\tau^k} = 0 \tag{4.3}$$

From (4.3) we obtain the recurrence relation

$$\frac{d^{k+1}y}{d^k y d\tau} = -\left(\frac{i\alpha\tau}{1 + i\alpha k\tau} + \frac{1}{1 + i\alpha k\tau} \frac{d^{k+2}y}{d^{k+1}y\, d\tau}\right)^{-1} \qquad (k = 0, 1, 2, \ldots) \tag{4.4}$$

Applying (4.4) successively to eliminate the differentials we obtain the following continued fraction for (4.1):

$$u^\circ(\tau) = \left[\frac{(i\alpha\tau)^{-1}}{1}, \quad \frac{[1 + (\nu - 1)\alpha i]\, \alpha^{-2}\tau^{-2}}{1}\right]_{\nu=2}^{\infty} \tag{4.5}$$

The convergence of the fraction (4.5) is not known, but a direct substitution shows that the convergents $u_k(\tau)$, where

$$u_1(\tau) = (i\alpha\tau)^{-1}, \qquad u_2(\tau) = \frac{(i\alpha\tau)^{-1}}{1 + (1 + i\alpha)\alpha^{-2}\tau^{-2}} \tag{4.6}$$
$$u_3(\tau) = \frac{(i\alpha\tau)^{-1}}{1 + \dfrac{(1 + i\alpha)\alpha^{-2}\tau^{-2}}{1 + (1 + 2i\alpha)\alpha^{-2}\tau^{-2}}}, \ldots$$

satisfy Equation (1.8) to the accuracy of the order $O(\alpha^{-k}\tau^{-2k})$. The continued fraction $u^\circ(\tau)$ of (4.5) tends asymptotically to a particular solution of Equation (1.8) as $\tau \to \infty$. Expanding this in a series of negative powers of τ we find that

$$u^\circ(\tau) = \frac{1}{i\alpha\tau} - \frac{1 + i\alpha}{i\alpha^2\tau^3} + \frac{(1 + i\alpha)(2 + 3i\alpha)}{i\alpha^5\tau^5} + O\left(\frac{1}{\alpha^4\tau^7}\right) \tag{4.7}$$

The solution $u^\circ(\tau) \to 0$ as $\tau \to \infty$, i.e. as $t \to \infty$. From (1.7) we can obtain an expression for a particular solution for z

$$z^\circ(t) = \exp\left(i\frac{C-A}{A}\int_0^t \omega_3\, dt - i\, Arg\,(w_2{}^\circ + i\omega_1{}^\circ)\right) u^\circ(\tau) \qquad (4.8)$$

Since $z^\circ(t) \to 0$ as $t \to \infty$ there exists a fixed vector γ° to which the ζ-axis tends as $t \to \infty$. The complex variable $z^\circ(t)$ determines the vector $-\gamma^\circ$. The vector γ° itself describes a ruled surface in the moving system of coordinates $\xi\eta\zeta$. It rotates about the ζ-axis with an angular velocity of approximately $(C-A)A^{-1}\omega_3$ and simultaneously approaches this axis.

We introduce a system of coordinates $\xi'\eta'\zeta$ which moves relative to the body and which is rotated about the ζ-axis through an angle φ relative to the $\xi\eta\zeta$-system, where

$$\varphi = \frac{C-A}{A}\int_0^t \omega_3\, dt - Arg\,(\omega_2{}^\circ - i\omega_1{}^\circ) \qquad (4.9)$$

In the $\xi'\eta'\zeta$-system the motion of the vector $-\gamma^\circ$ is described by the complex variable $u^\circ(\tau)$ which varies only slightly for sufficiently large values of $t > 0$. Consequently the $\xi'\eta'\zeta$-system rotates about the vector γ° with an angular velocity which proves to be approximately equal to $CA^{-1}\omega_3$.

Finally, for large values of $t > 0$ the motion has the following properties. There exists a fixed vector γ° which makes a continuously diminishing angle

$$A\,|\,\omega_1{}^\circ + i\omega_2{}^\circ\,|\,m^{-1}t^{-1} + O(t^{-2})$$

with the ζ-axis. The body rotates with an angular velocity

$$(A - C)\,A^{-1}\,mt + O(1)$$

about the ζ-axis. The ζ-axis rotates about the vector γ° at an angular velocity $CA^{-1}\,mt + O(1)$.

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